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Arbitrary Wave Relativistic Bound State Solutions for the Eckart Potential

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Abstract The approximately analytical bound state solutions of the l-wave Klein-Gordon and k-state Dirac equations with the mixed Eckart potentials are carried out by a proper approximation to the centrifugal term. The analytical radial wave functions of the l-wave Klein-Gordon and k-state Dirac equations with the mixed Eckart potentials are presented and the corresponding energy equations are derived. Two special cases for $k = 1$ and for $k = 1$ and $\beta = 0$ are studied briefly. Finally, we also verify the rationality of this approximation.

Keywords Eckart potential · Klein-Gordon and Dirac equations · Approximately analytical solution

1 Introduction

It is well known that the exact solutions play an important role in quantum mechanics since they contain all the necessary information regarding the quantum system under consideration. For example, the exact solutions of the Schrödinger equation for a hydrogen atom and for a harmonic oscillator in three dimensions are an important milestone at the beginning stage of quantum mechanics, which provided a strong evidence for supporting the correctness of the quantum theory [1–3]. However, the exact solutions are few so that many quantum systems have to be treated by approximate methods. On the other hand, recently,

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there has been a growing interest in searching for the analytical solutions with the framework of non-relativistic and relativistic quantum mechanics [4–12]. In the setting of the Klein-Gordon and Dirac equations, some authors have assumed that the scalar potential is equal to the vector potential in order to analytically solve the Klein-Gordon and Dirac equations for some potential such as Rosen-Morse-type potential [10], Scarf-type potential [12], Eckart potential [13, 14], Manning-Rosen potential [15], and Hulthén potential [16, 17], etc. However, it should be mentioned that almost these contributions mentioned above are made to s-wave case. Nevertheless, we have to study both of them in order to understand the studied quantum system completely. Based on the previous work [14, 15], we have applied a proper approximation to the centrifugal term and obtained approximately analytical l-wave scattering solutions of the Schrödinger equation with the Manning-Rosen potential [18], the Eckart potential [19] and approximately analytical bound state solutions of the l-wave Klein-Gordon equation with the Manning-Rosen potential [20]. Motivated by recent work about the scattering states of the l-wave Schrödinger equation with the Eckart potential carried out by one of the present authors, Wei *et al.* [19], in this work, we attempt to study the relativistic bound state solutions of the l-wave Klein-Gordon and k-state Dirac equations with the mixed Eckart potentials, which is given by [14, 19]

$$V(r) = -\alpha \frac{e^{-r/a}}{1-e^{-r/a}} + \beta \frac{e^{-r/a}}{(1-e^{-r/a})^2}, \quad \alpha, \beta > 0, \tag{1}$$

where parameters α and β describe the depth of potential well, and parameter a is related to the range of the potential. The reader can refer to [13, 14, 19, 21] for more information and possible applications of this potential.

The purpose of this work is devoted to studying the analytical bound state solutions of the arbitrary l-wave Klein-Gordon and k-state Dirac equations with the mixed Eckart potentials, since theoretical prediction of many properties of molecule model in physics and chemical physics requires the knowledge of the radial wave functions of l-wave bound states. This problem has not been reported by far to our knowledge.

This paper is organized as follows. In Sect. 2 and 3 the relativistic bound state solutions of the l-wave Klein-Gordon and k-state Dirac equations with the mixed Eckart potentials are presented, respectively. In Sect. 4 two special cases for $k = 1$ and for $k = 1$ and $\beta = 0$ are also studied briefly. The conclusions are given in Sect. 5.

2 Arbitrary l-wave Bound State Solution of the Klein-Gordon Equation

In spherical coordinates, the Klein-Gordon equation with the scalar potential $S(r)$ and vector potential $V(r)$ is given by ($\hbar = c = 1$)

$$\{-\nabla^2 + [M + S(r)]^2\}\psi(r,\theta,\phi) = [E - V(r)]^2\psi(r,\theta,\phi). \tag{2}$$

By taking $\psi(r,\theta,\phi) = r^{-1}u(r)Y_{lm}(\theta,\phi)$, and substituting it into (2), we obtain the radial Klein-Gordon equation as

$$\frac{d^2u(r)}{dr^2} + \left\{[E^2 - M^2] - 2[MS(r) + EV(r)] + [V^2(r) - S^2(r)] - \frac{l(l+1)}{r^2}\right\}u(r) = 0. \tag{3}$$

Here, we consider the case that the scalar potential and vector potential are equal Eckart potential, i.e.

$$S(r) = V(r) = -\alpha \frac{e^{-r/a}}{1-e^{-r/a}} + \beta \frac{e^{-r/a}}{(1-e^{-r/a})^2}, \tag{4}$$

and inserting it into (3), we have

$$\frac{d^2u(r)}{dr^2}+\left\{[E^2-M^2]-2(M+E)\left[-\alpha\frac{e^{-r/a}}{1-e^{-r/a}}+\beta\frac{e^{-r/a}}{(1-e^{-r/a})^2}\right]-\frac{l(l+1)}{r^2}\right\}u(r)=0. \tag{5}$$

It is obvious to show that (5) cannot be solved analytically except for $l=0$ case [14, 15]. Therefore, we must use a proper approximation to centrifugal term similar to other authors [14, 15, 18–20]. It is noted that for short potential range (namely, bigger parameter a) the following formula[1]

$$\frac{1}{r^2}\approx\frac{e^{-r/a}}{a^2(1-e^{-r/a})^2}, \tag{6}$$

is a good approximation to $1/r^2$. Inserting this into (5) allows us to obtain

$$\frac{d^2u(r)}{dr^2}+\left\{[E^2-M^2]-2(M+E)\left[-\alpha\frac{e^{-r/a}}{1-e^{-r/a}}+\beta\frac{e^{-r/a}}{(1-e^{-r/a})^2}\right]-\frac{l(l+1)e^{-r/a}}{a^2(1-e^{-r/a})^2}\right\}u(r)=0. \tag{7}$$

Introducing a new variable $x=e^{-r/a}$, (7) can be further transformed into the following form

$$x^2\frac{d^2u(x)}{dx^2}+x\frac{du(x)}{dx}-\left[\lambda^2-\frac{k^2\alpha x}{1-x}+\frac{k^2\beta x}{(1-x)^2}+\frac{l(l+1)x}{(1-x)^2}\right]u(x)=0, \tag{8}$$

where $\lambda^2=a^2(M^2-E^2)$, $k^2=2(M+E)a^2$.

According to Frobenius theorem, the singularity points of the above differential equation play an important role in the expression of the wave functions. The singular points here are at $x=0$ and at $x=1$. As a result, we take wave function of the form

$$u(x)=x^{\lambda}(1-x)^{1+\delta}f(x), \tag{9}$$

where

$$\delta=\frac{1}{2}\left[-1+\sqrt{4k^2\beta+4l(l+1)+1}\right]. \tag{10}$$

Inserting (9) into (8), we have

$$x(1-x)\frac{d^2}{dx^2}f(x)+[2\lambda+1-(2\lambda+2\delta+3)x]\frac{d}{dx}f(x)+[k^2\alpha-k^2\beta-(1+\delta)(2\lambda+1)-l(l+1)]f(x)=0, \tag{11}$$

whose solutions are nothing but hypergeometric functions [24]

$$f(x)={}_2F_1(\gamma;\mu;\nu;x)=\sum_{q=0}^{\infty}\frac{(\gamma)_q(\mu)_q}{(\nu)_q}\frac{x^q}{q!}, \tag{12}$$

[1] This approximation was first introduced by Greene and Aldrich in order to generate pseudo-Hulthén wave functions for $l\neq 0$ states [22]. By this approximate method, the authors Dong et al. [14] have calculated the bound state energy levels for this potential and presented the numerical results for arbitrary quantum number n,l and a given a. It is shown that the approximately analytical results are in better agreement with the previous work by Lucha and Schöberl [23] for short potential range.

where

$$(x)_q = \frac{\Gamma(x+q)}{\Gamma(x)}, \qquad \gamma = \lambda + \delta + 1 - \xi,$$
$$\mu = 1 + \lambda + \delta + \xi, \qquad \nu = 2\lambda + 1, \tag{13}$$
$$\xi = \sqrt{\lambda^2 + \delta^2 + \delta + k^2\alpha - k^2\beta - l(l+1)},$$

and for the bound states, the hypergeometric functions must be terminated with a polynomial [25], which demands

$$\gamma = \lambda + \delta + 1 - \xi = -n \quad (n = 0, 1, 2, \ldots), \tag{14}$$

with which and (13), the energy equation can be expressed as the following forms

$$\frac{1}{2} + n + a\sqrt{M^2 - E^2} + \sqrt{2a^2\beta(M+E) + l(l+1) + \frac{1}{4}} - a\sqrt{M^2 - E^2 + 2\alpha(M+E)} = 0$$
$$(n = 0, 1, 2, \ldots,\; l = 0, 1, 2, \ldots). \tag{15}$$

From (9) and (12), the correspondingly radial wave functions of arbitrary l-wave bound states of the Klein-Gordon equation with the mixed Eckart potentials can be written as

$$u(r) = (e^{-r/a})^{\lambda}(1 - e^{-r/a})^{1+\delta}{}_2F_1(-n; 2\lambda + 2\delta + 2 + n; 2\lambda + 1; e^{-r/a}), \tag{16}$$

where δ and λ are determined by (10) and (8).

3 Arbitrary *k*-state Bound State Solution of the Dirac Equation

According to [4, 13], the Dirac equation with both scalar potential $S(r)$ and vector potential $V(r)$ can be written as ($\hbar = c = 1$)

$$\{\boldsymbol{\alpha} \cdot \mathbf{p} + \beta[M + S(r)]\}\psi(r) = [E - V(r)]\psi(r). \tag{17}$$

In the relativistic quantum mechanics, the complete set of the conservative quantities for a particle in a central field can be taken as (H, K, J^2, J), the spinor eigenfunctions of which are given by

$$\psi = \frac{1}{r}\begin{pmatrix} f(r)\phi^A_{jm_j} \\ ig(r)\phi^B_{jm_j} \end{pmatrix} \quad \left(k = j + \frac{1}{2}\right), \tag{18a}$$

$$\psi = \frac{1}{r}\begin{pmatrix} f(r)\phi^B_{jm_j} \\ ig(r)\phi^A_{jm_j} \end{pmatrix} \quad \left(k = -j - \frac{1}{2}\right), \tag{18b}$$

where

$$\phi^A_{jm_j} = \begin{pmatrix} \sqrt{\frac{j+m_j}{2j}}Y_{j-\frac{1}{2},m_j-\frac{1}{2}} \\ \sqrt{\frac{j-m_j}{2j}}Y_{j-\frac{1}{2},m_j+\frac{1}{2}} \end{pmatrix}, \qquad \phi^B_{jm_j} = \begin{pmatrix} -\sqrt{\frac{j-m_j+1}{2j+2}}Y_{j+\frac{1}{2},m_j-\frac{1}{2}} \\ \sqrt{\frac{j+m_j+1}{2j+2}}Y_{j+\frac{1}{2},m_j+\frac{1}{2}} \end{pmatrix},$$

$f(r)$ and $g(r)$ are real radial square-integral functions. The spin-orbit matrix operator K is defined as $\hbar K = \begin{pmatrix} \hbar+\sigma\cdot\mathbf{l} & 0 \\ 0 & -\hbar-\sigma\cdot\mathbf{l} \end{pmatrix}$, and the total angular momentum operator is $\mathbf{J} = \mathbf{l} + \frac{\hbar}{2}\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$, where σ denotes the vector Pauli spin matrix and $\mathbf{l}$ denotes orbit angular momentum operator. For a given total angular momentum j, the spin-orbit matrix quantum number k is taken as $k = j + \frac{1}{2}$ if the orbit angular momentum $l = j - \frac{1}{2}$, and $k = -(j + \frac{1}{2})$ if $l = j + \frac{1}{2}$, and m stands for the integers in the range $j, j-1, \ldots, -j$.

Substituting (18a) or (18b) into (17), we obtain the radial Dirac equation as

$$\frac{df}{dr} - \frac{k}{r}f = [M + E + S(r) - V(r)]g, \tag{19a}$$

$$\frac{dg}{dr} + \frac{k}{r}g = [M - E + S(r) + V(r)]f. \tag{19b}$$

In the case of equal scalar and vector Eckart potentials, i.e., $V(r) = S(r)$, (19a) and (19b) can be simplified as the following two coupled differential equations

$$\frac{df}{dr} - \frac{k}{r}f = (M + E)g, \tag{20a}$$

$$\frac{dg}{dr} + \frac{k}{r}g = [M - E + 2V(r)]f. \tag{20b}$$

Substituting (20a) into (20b), we obtain a Schrödinger-like equation for the arbitrary spin-orbit coupling quantum number k as follows

$$\frac{d^2 f(r)}{dr^2} + \left\{[E^2 - M^2] - 2(M+E)\left[-\alpha\frac{e^{-r/a}}{1-e^{-r/a}} + \beta\frac{e^{-r/a}}{(1-e^{-r/a})^2}\right] - \frac{k(k-1)}{r^2}\right\} f(r) = 0. \tag{21}$$

It is obvious to show that (21) is the same as (7). As a result, the bound state energy equation of arbitrary k-state Dirac equation with the mixed Eckart potentials can be written as

$$\begin{aligned} &\frac{1}{2} + n + a\sqrt{M^2 - E^2} + \sqrt{2a^2\beta(M+E) + k(k-1) + \frac{1}{4}} \\ &\quad - a\sqrt{M^2 - E^2 + 2\alpha(M+E)} = 0 \quad (n = 0, 1, 2, \ldots, k = 1, 2, \ldots). \end{aligned} \tag{22}$$

The corresponding upper spinor wave functions $f(r)$ can be given by

$$f(r) = (e^{-r/a})^{\lambda}(1 - e^{-r/a})^{1+\delta}{}_2F_1(-n; 2\lambda + 2\delta + 2 + n; 2\lambda + 1; e^{-r/a}). \tag{23}$$

The substitution of (23) into (20a) gives the correspondingly low spinor wave functions $g(r)$ as

$$\begin{aligned} g(r) = &-\frac{(e^{-r/a})^{\lambda}(1-e^{-r/a})^{\delta}}{M+E}\left\{\left[\frac{\lambda(1-e^{-r/a})}{a} - \frac{(1+\delta)e^{-r/a}}{a} + \frac{k}{r}(1-e^{-r/a})\right]\right. \\ &\times {}_2F_1(-n; 2\lambda + 2\delta + 2 + n; 2\lambda + 1; e^{-r/a}) - \frac{(2\lambda + 2\delta + 2 + n)n}{a(2\lambda + 1)}e^{-r/a}(1 - e^{-r/a}) \\ &\left.\times {}_2F_1(-n + 1; 2\lambda + 2\delta + 3 + n; 2\lambda + 2; e^{-r/a})\right\}. \end{aligned} \tag{24}$$

Inserting $f(r)$ and $g(r)$ into (18), we can obtain the bound state spinor wave functions of the Dirac equation with the mixed Eckart potentials for arbitrary spin-orbit coupling quantum number k.

4 Discussion

In this section we study two special cases. First, let us study the case $k=1$. From (22), the energy equation of the Dirac equation with mixed Eckart potential are simplified as

$$\frac{1}{2}+n+a\sqrt{M^2-E^2}+\sqrt{2a^2\beta(M+E)+\frac{1}{4}}-a\sqrt{M^2-E^2+2\alpha(M+E)}=0. \quad (25)$$

If considering the Eckart potential with a constant shift $-V_2$, i.e.

$$V(r)=-\alpha\frac{e^{-r/a}}{1-e^{-r/a}}+\beta\frac{e^{-r/a}}{(1-e^{-r/a})^2}-V_2, \quad (26)$$

and comparing it with (21) of [13], and observing the relations of corresponding parameters, then (25) can be rearranged as

$$M^2-E^2=\frac{(M+E)^2V_2^2}{\alpha^2}\frac{1}{(n+\eta)^2}+\alpha^2(n+\eta)^2, \quad (27)$$

where

$$\eta=\frac{1}{2}\left[1+\sqrt{1+8a^2\beta(M+E)}\right], \quad (28)$$

which is consistent with the exact energy levels of the s-wave bound states of the Dirac equation with the mixed Eckart potentials of [13].

Second, we study the case $k=1$ and $\beta=0$. Obviously, the Eckart potential in this case reduces to the Hulthén potential $V(r)=-\alpha e^{-r/a}/(1-e^{-r/a})$. If so, the energy equation (22) reduces to the exact energy levels of the s-wave bound states of the Dirac equation with Hulthén potential as follows

$$M^2-E^2=\left[\frac{a\alpha(M+E)}{1+n}+\frac{1+n}{2a}\right]^2, \quad (29)$$

which is consistent with that given in (27) of [13].

5 Conclusions

In this work the approximately analytical bound state solutions of the l-wave Klein-Gordon and k-state Dirac equations with the mixed Eckart potentials have been presented by taking a proper approximation scheme to the centrifugal term. The analytically radial wave functions and corresponding energy equations of the l-wave Klein-Gordon and k-state Dirac equations are obtained. Also, we have considered and verified two special cases for $k=1$ and for $k=1$ and $\beta=0$. It is found that these obtained results reduce to those of s-wave bound states of the Eckart potential and the Hulthén potential, respectively. Before ending this work, we give some useful remarks on this approximation. First, we have applied this approximation to study the arbitrary l-wave scattering state solutions of the Schrödinger equations for Manning-Rosen and Eckart potentials, respectively. It is shown that this approximation is a good approximation to centrifugal term for short potential range (namely, bigger parameter a). Second, in order to further show the differences between the centrifugal term and this approximation as well as the effects of parameter a on this approximation, the differences

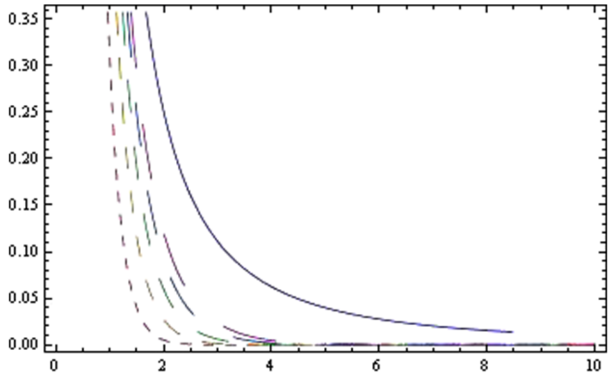

Fig. 1 The centrifugal term $\frac{1}{r^2}$ and the proper approximation to it $\frac{e^{-r/a}}{a^2(1-e^{-r/a})^2}$ as the functions of the variable r with parameters $a = 0.25, 0.35, 0.45, 0.55, 0.65$ are displayed from the *left* side to the *right*

between the centrifugal term and this approximation to it are shown in Fig. 1. The differences between them are displayed from left side to the right with $a = 0.25, 0.35, 0.45, 0.55, 0.65$. It is shown that the differences between them are small and decreasing with the increasing parameter a, so this approximation is an effective approximation to centrifugal term for short potential range (namely, bigger parameter a). However, the differences between them will appear for small values of the parameter a. This means that this approximation is not a good approximation to a centrifugal term when the potential parameter a becomes small. Finally, it should be mentioned that another approximation [15] $\frac{1}{r^2} \approx \frac{1}{a^2}\frac{e^{-2r/a}}{(1-e^{-r/a})^2}$ is not as good as the present one (6) since the series expansion of the expression $\frac{1}{a^2}\frac{e^{-2r/a}}{(1-e^{-r/a})^2} \approx \frac{1}{r^2} - \frac{1}{ar}$ includes a Coulomb term.

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